A STEFAN PROBLEM WITH A BOUNDARY CONDITION OF THE FOURTH KIND

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We reduce a Stefan problem with a boundary condition of the fourth kind, using the method of quasi-stationary states, to the solution of a system of two first-order ordinary differential equations in the unknowns of the boundary separating the phases and the boundary temperature.

In calculating the work cycles of underground reservoirs of condensed gases there is considerable interest in the temperature variation dynamics inside the reservoir.

If such a reservoir is situated in a water-saturated soil, then after it is filled, the soil begins to freeze over and the temperature of the compressed gas begins to increase (see [1]). The following two problems are of interest from the point of view of rational operation of the reservoir: a) After how long a time t_g is the temperature $T_\Gamma(t)$ of the compressed gas raised from its initial temperature T_i to some temperature T_g, defined by technological considerations? b) What must be the thermal resistance of the reservoir walls so that after the given time t_g the temperature $T_\Gamma(t)$ will not have risen above a permissible level?

We assume that the reservoir is a sphere of radius R_0. Then we formulate the following problem for the determination of $T_g(t)$.

Suppose that the region exterior to the sphere of radius R_0 contains water-saturated soil with temperature $T_0 > 0$. Let the interior of the sphere be occupied by a well mixed liquid, which has at the time $t = 0$ a temperature $T_i < 0$. We assume also that this sphere has a known thermal resistance. It is obvious that freezing of the soil around the sphere commences at some instant $t = t^*$. We determine the law of motion for the boundary $R_p(t)$ separating the phases and the temperature variation of the liquid inside the sphere. To do this we need to solve a Stefan problem having the following boundary and initial conditions [2]:

$$\begin{gathered}
\frac{\partial T_1}{\partial t} = \varkappa_1 \left(\frac{\partial^2 T_1}{\partial r^2} + \frac{2}{r} \frac{\partial T_1}{\partial r} \right) \quad (R_0 \leqslant r \leqslant R_p(t)) \\
\frac{\partial T_2}{\partial t} = \varkappa_2 \left(\frac{\partial^2 T_2}{\partial r^2} + \frac{2}{r} \frac{\partial T_2}{\partial r} \right) \quad (R_p(t) \leqslant r < \infty) \\
\lambda_1 \iint\limits_{S(R_0)} \frac{\partial T_1}{\partial r} dS + M_g c_g \frac{dT_\Gamma}{dt} = 0 \quad (r = R_0) \\
\partial T_1 / \partial r - h_1 (T_1 - T_\Gamma) = 0 \quad (r = R_0) \\
T_1(R_p, t) = T_2(R_p, t) = T_p \quad (r = R_p(t)) \\
\lambda_2 \frac{\partial T_2}{\partial r} - \lambda_1 \frac{\partial T}{\partial r} = L \rho_2 W \frac{dR_p(t)}{dt} \quad (r = R_p(t)) \\
T_2(r, 0) = T_0
\end{gathered} \tag{1}$$

Here T is the temperature, t is the time, r is a variable radius, R_0 is the sphere radius, R_p is the coordinate of the boundary separating the phases; $\varkappa$, λ, c, and ρ are, respectively, the thermal diffusivity coefficient, the thermal conductivity coefficient, the specific heat, and the density; M_g is the mass of the compressed gas; $S(R_0)$ is the surface of the sphere; T_0 is the initial soil temperature and T_p is the phase change temperature; L is the latent heat of the phase transition; W is the moisture content, the subscripts 1 and 2 referring, respectively, to the frozen and thawed zones; the subscript g is attached to the compressed gas parameters.

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It is well known (see [3]) that for stationary boundary conditions of the first and third kinds the small rate of movement of the boundary separating the phases makes it possible to usefully employ the quasistationary method (see [4]). The slow rate of movement of the boundary stipulates a slow change in the temperature of the compressed gas inside the sphere, since an arbitrary isotherm $T < 0$ cannot fit the isotherm $T = T_p = 0$ (T_p is the phase transition temperature from water to ice).

Then, following the method referred to above, we have

$$T_1(r) = T_p - \frac{h_1 R_0^2 (T_p - T_\Gamma)}{[h_1 R_0 (R_p - R_0) - R_p]} \left(\frac{R_p}{r} - 1 \right)$$
$$T_2(r, t) = T_0 - \frac{(T_0 - T_p)}{r} R_p \operatorname{erfc} \left[\frac{r - R_p}{2\sqrt{\varkappa_2 t}} \right] \quad (2)$$

The third and sixth conditions in Eq. (1), together with the Eqs. (2), yield a system of two ordinary differential equations in $R_p(t)$ and $T_g(t)$.

In the dimensionless variables

$$\vartheta = \frac{T}{T_i}, \quad \xi = \frac{R_p}{R_0}, \quad \tau = \frac{t\varkappa_1}{R_0^2}, \quad h = h_1 R_0, \vartheta_g \frac{T_\Gamma}{T_i}$$

this system has the form

$$\frac{d\vartheta_g}{d\tau} = \frac{3c_1\rho_1}{c_g\rho_g} \frac{h(\vartheta_p - \vartheta_g)\xi}{[h(\xi - 1) + \xi]}$$
$$\frac{d\xi}{d\tau} = \frac{c_1\rho_1 T_i}{L\rho_2 W} \frac{h(\vartheta_p - \vartheta_g)}{[h(\xi - 1) + \xi]\xi} - \frac{\varkappa_2}{\varkappa_1} \frac{c_2 T_i (\vartheta_0 - \vartheta_p)}{LW} \left[\frac{1}{\xi} + \sqrt{\frac{\varkappa_1}{\pi\varkappa_2\tau}} \right] \quad (3)$$

For assigned initial conditions it is necessary to solve the corresponding problem without phase transitions and to determine $\tau^* = t^*\varkappa_1/R_0^2$ and $\varkappa_g(\tau^*) = T_g(t^*)/T_i$ from the condition of equality of the temperature on the outer boundary of the sphere to the phase transition temperature.

The initial conditions will then have the form

$$\vartheta_g(\tau^*) = \vartheta_g, \quad \xi(\tau^*) = 1 \quad (4)$$

The system (3) with the initial conditions (12) can be solved on an electronic digital computer using standard procedures.

The method of solving problem a) is obvious. Knowing the solution of the system (3) with the initial conditions (4) for $\vartheta_g(\tau)$, we determine the time $\tau_g = t_g\varkappa_1/R_0^2$ from the condition

$$\vartheta_g(\tau_g) = \vartheta_{gi} \quad (\vartheta_{gi} = T_g/T_i)$$

If we neglect the thermal resistance of the hull, then in place of the initial conditions (4) we shall have

$$\vartheta_g(0) = 1, \quad \xi(0) = 1 \quad (5)$$

which simplifies our problem since the parameters determined from the problem without phase transition all vanish. The system corresponding to this case is obtained from Eqs. (3) by letting $h \to \infty$.

If we neglect the heat flow from the thawed zone, which in the majority of practically important cases is small by virtue of the insignificant difference of the initial soil temperature and the phase transition temperature, then from the system (3) we obtain

$$\frac{d\vartheta_g}{d\tau} = \frac{3c_1\rho_1}{c_g\rho_g} \frac{h(\vartheta_p - \vartheta_g)\xi}{[h(\xi - 1) + \xi]} \quad (6)$$

$$\frac{d\xi}{d\tau} = \frac{c_1\rho_1 T_i}{L\rho_2 W} \frac{h(\vartheta_p - \vartheta_g)}{[h(\xi - 1) + \xi]\xi} \quad (7)$$

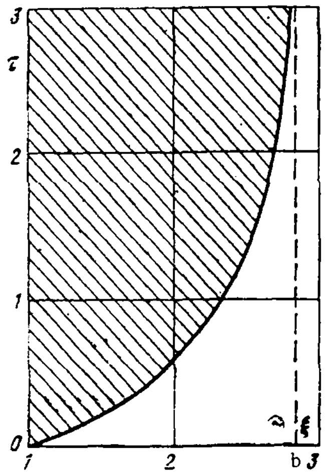

Fig. 1

The initial conditions for this system are given by equations (5). The solution of the system (6), (7) may be found in terms of quadratures. Dividing Eqs. (6) by Eq. (7), we have

$$\frac{d\vartheta_g}{d\xi} = \frac{3L\rho_2 W}{c_g \rho_g T_i}\xi^2 \tag{8}$$

$$\vartheta_g(1) = 1 \tag{9}$$

From this it follows that

$$\vartheta_g(\xi) = L\rho_2 W (c_g \rho_g T_i)^{-1}(\xi^3 - 1) + 1 \tag{10}$$

We note from Eq. (8) that the curves $\vartheta_g(\xi)$ are independent of the parameter h. Physically this means that for an arbitrary thermal resistance there corresponds to a definite temperature inside the sphere a definite position of the boundary separating the phases. The role of h is then reduced merely to influencing the rate at which this state is attained.

Substituting Eq. (10) into Eq. (7) and carrying out the integration, we obtain

$$\tau = \frac{1}{3b\alpha H}\left\{(b - H)\ln\frac{b-1}{b-\xi} - \left(b + \frac{H}{2}\right)\ln\frac{b^2 + b\xi + \xi^2}{b^2 + b + 1} + \sqrt{3}\, H \operatorname{arc\,tg}\frac{\sqrt{3}\, b(\xi - 1)}{2b^2 + b + b\xi + 2\xi}\right\}$$
$$\left(\alpha = \frac{c_1\rho_1}{c_g\rho_g},\ H = \frac{h}{h+1},\ b^3 = 1 + \frac{(\vartheta_p + 1)\, c_g\rho_g T_i}{L\rho_2 W}\right) \tag{11}$$

For problem a) we obtain the desired time τ_g from Eq. (11) by substituting ξ_g into the right hand side for ξ; ξ_g is obtained uniquely from Eq. (10) in terms of the assigned temperature ϑ_{gi}.

Under these same assumptions we can also find the solution of problem b) in explicit form.

To do this we solve Eq. (11) for h, wherein all the insulation parameters make their appearance:

$$h = \left[b\ln\frac{b^3 - 1}{b^3 - \xi}\right]\left\{3b\alpha\tau - \sqrt{3}\operatorname{arc\,tg}\left[\frac{\sqrt{3}\, b(\xi - 1)}{2b^2 + b + b\xi + 2\xi}\right] - \right.$$
$$\left. - \ln\frac{(b^3 - 1)^b (b^2 + b + 1)^{3/2}}{(b^3 - \xi^3)^b (b^2 + b\xi + \xi^2)^{3/2}}\right\}^{-1} \tag{12}$$

Substituting τ_g in place of τ and ξ_g, obtained from Eq. (10) for a known value of ϑ_g, in place of ξ, we find the necessary value of h.

If the denominator in Eq. (12) vanishes, then $h = \infty$, i.e., the curve

$$3b\alpha\tau - \sqrt{3}\operatorname{arc\,tg}\left[\frac{\sqrt{3}\, b(\xi - 1)}{2b^2 + b + b\xi + 2\xi}\right] - \ln\left[\frac{(b^3 - 1)^b (b^2 + b + 1)^{3/2}}{(b - \xi)^b (b^2 + b\xi + \xi^2)^{3/2}}\right] = 0$$

shown in the figure, is the solution $\xi(\tau)$ of the system (6), (7) in the absence of insulation.

The quantity b here is the dimensionless limiting radius for the boundary of freezing.

The problem in which insulation is needed is solved by having the point (τ_g, ξ_g) fall into the shaded region of the figure. When the point lies outside this region the conditions of problem b) are satisfied, also without insulation.

LITERATURE CITED

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